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Equivariant BRST quantization and reducible symmetries

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Abstract
Working from first principles, quantization of a class of Hamiltonian systems with reducible symmetry is carried out by constructing first the appropriate reduced phase space and then the BRST cohomology. The constraints of this system correspond to a first class set for a group G and a second class set for a subgroup H. The BRST operator constructed is equivariant with respect to H. Using algebraic techniques analogous to those of equivariant de Rham theory, the BRST operator is shown to correspond to that obtained by BV quantization of a class of systems with reducible symmetry. The 'ghosts for ghosts' correspond to the even degree two generators in the Cartan model of equivariant cohomology. As an example of the methods developed, a topological model is described whose BRST quantization relates to the equivariant cohomology of a manifold under a circle action.

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1. Introduction

In this paper, we derive from first principles a BRST procedure for quantization of certain symmetric Hamiltonian systems for which the constraints do not form a closed algebra, because the symmetries are what is known as reducible. It is shown that the resulting BRST operator compares to the standard BRST operator for the related irreducible symmetry in the same way that in equivariant de Rham cohomology the equivariant derivative compares to the standard exterior derivative. The auxiliary even generators which occur in this equivariant cohomology correspond to the 'ghosts for ghosts' in the BV quantization of reducible symmetries first formulated by Batalin and Vilkovisky [1, 2].

The underlying motivation for this work is a desire to understand the functional integral methods which have proved so powerful in the quantization of theories with symmetry. An example is constructed showing that the procedure leads to a full path integral quantization scheme complete with a quantum gauge-fixing procedure, so that quantum calculations are possible.

In a standard symmetric Hamiltonian system, there is a set of 'first class' constraints $T_a, a = 1, \ldots, m$ on the phase space of the system which are closed under Poisson bracket:

$$\{T_a, T_b\} = C_{ab}^c T_c.$$

The constraints are a reflection of the symmetry of the system under some group G, and the true phase space of the system is the quotient of the constrained surface by the group action. (A more intrinsic, group theoretic formulation is described below in section 2.) In the simplest situation the coefficients C_{ab}^c are constants and the constraint algebra is a finite-dimensional Lie algebra, but it is often the case that the coefficients C_{ab}^c are more general functions on the phase space, although the system may still possess symmetry related to a finite-dimensional Lie group, as explained in appendix A. There are however symmetric systems for which the constraint algebra does not close, with some of the constraints being second class. The purpose of this paper is to show that for a class of such systems there is an analysis in terms of a symmetry group G acting equivariantly with respect to a subgroup H, and to derive the corresponding BRST quantization scheme. The result is that the true phase space of the system is obtained in two stages, first reducing the phase space by the action of H and then by G. (Such two stage reduction is described extensively by Marsden, Misiolek, Ortega, Perlmutter and Ratiu in [25, 26].)

The BRST operator obtained is equivalent to that used in BV quantization of first-order reducible systems [1–3], but the derivation is more fundamental, using the algebraic features of the constraints to construct the appropriate reduced phase space. The even, ghost number two, fields of the BV formalism correspond to the degree two generators of the dual of the Lie algebra of H in the Weil model of H-equivariant cohomology. The relation between equivariant cohomology and BRST quantization of certain topological theories has been pointed out by a number of authors, including Kalkman, Chemla and Kalkman and Ouvry, Stora and van Baal [4–6]. In this paper, we take these ideas further and give general arguments based on canonical quantization and the necessary modification of the Marsden–Weinstein reduction process [7] for the open constrained systems studied, obtaining both a more general and a more fundamental explanation of this connection.

The structure of the paper is that in section 2 we first describe the symplectic geometry of a standard constrained system with closed algebra, including the moment map and the Marsden–Weinstein reduction procedure [7] leading to the reduced phase space of the system obtained after gauge redundancy has been removed. We then describe the more general systems considered in this paper, and the corresponding reduced phase space. (In appendix A, which relates to systems with closed constraint algebras as well as the more general systems considered in this paper, we explain that the reduction process—and hence the BRST procedure—are also applicable in the setting of a more general class of group action, in which an infinite-dimensional group acts on the phase space, with a related action of the finite-dimensional group G which is only local; this allows for the possibility of 'structure functions' and apparent variations in the constraint algebra.) In section 3 the standard Hamiltonian BRST method, due to Henneaux [8], Kostant and Sternberg [9] and Stasheff [31] is reviewed, while in section 4 the equivariant BRST operator appropriate for the more general constrained system described in section 2 is constructed. Using methods adapted from equivariant de Rham theory, various different but equivalent models of the BRST cohomology are presented. The class of reducible symmetries which leads to the constrained systems studied in this paper is described in section 5. In the final section a specific model exhibiting these structures is described. The BRST operator is that constructed by Kalkman [4] and by Chemla and Kalkman [5], but our derivation is from a simple classical action. The model itself is equivalent to the supersymmetric model introduced by Witten [10] and used by Witten and others to obtain powerful equivariant localization techniques, as may be seen for instance in [11, 13–16, 23].

2. The reduced phase space of a partly open constraint algebra

In this section we will consider the classical dynamics of a Hamiltonian system defined on a $2n$-dimensional symplectic manifold $\mathcal{N}$ on which an m-dimensional Lie group G acts freely and symplectically on the left, with $m \leqslant n$. To establish notation, gy will denote the image of the point y in $\mathcal{N}$ under the left action of g in G, and for each ξ in the Lie algebra $\mathfrak{g}$ of G, $\underline{\xi}$ will denote the corresponding vector field on $\mathcal{N}$. The group action is required to be Hamiltonian, so that there exists a map (referred to as the *constraint map*) $T : \mathfrak{g} \to \mathcal{F}(\mathcal{N}), \xi \mapsto T_\xi$ (where $\mathcal{F}(\mathcal{N})$ denotes the space of smooth functions on $\mathcal{N}$) which satisfies the conditions

$$\mathcal{L}_{\underline{\xi}} f = \{T_\xi, f\} \qquad T_\xi(gy) = T_{\mathrm{Ad}_g \xi}(y) \tag{1}$$

for all f in $\mathcal{F}(\mathcal{N})$, y in $\mathcal{N}$ and g in G [20]. Here $\{,\}$ denotes the Poisson bracket with respect to the symplectic form on $\mathcal{N}$. This is the standard set-up for a constrained Hamiltonian system: the constraint functions are the m functions $T_a \cong T_{\xi_a}$ corresponding to a basis $\{\xi_a | a = 1, \ldots, m\}$ of $\mathfrak{g}$, and the constraint submanifold C is the subset of $\mathcal{N}$ consisting of points y such that $T_a(y) = 0$ for $a = 1, \ldots, m$. More intrinsically, C is the set $\phi^{-1}(0)$, where $\phi : \mathcal{N} \to \mathfrak{g}^*$ is the moment map, which is the transpose of the constraint map T and thus defined by

$$\langle \phi(y), \xi \rangle = T_\xi(y) \tag{2}$$

for all y in $\mathcal{N}$ and ξ in $\mathfrak{g}$. By properties (1) of the map T, C is invariant under the action of G; the Marsden–Weinstein reduction theorem [7] states that the quotient manifold C/G is a symplectic manifold with a symplectic form ν determined uniquely by the condition $\pi^*\nu = \iota^*\omega$, where ω is the symplectic form on $\mathcal{N}$, $\iota : C \to \mathcal{N}$ is inclusion and $\pi : C \to C/G$ is the canonical projection. This result can be proved using theorem 25.2 of [20], which establishes that in certain circumstances if ω is a closed form on a manifold X then the set of vector fields $\underline{\xi}$ on X which satisfy

$$\iota_{\underline{\xi}}\omega = 0 \tag{3}$$

forms an integrable distribution, and the corresponding foliation is fibrating; further, if $\rho : X \to \mathcal{M}$ is the fibration, there is a symplectic form ν on $\mathcal{M}$ uniquely determined by $\omega = \rho^*(\nu)$. The symplectic manifold obtained by this two stage reduction process is referred to as the reduced phase space of the system and will be denoted $\mathcal{N}/\!\!/G$. It is the true phase space of the system; however it is in general a rather complicated space, even when $\mathcal{N}$ is simple, and may not admit a polarization as required in geometric quantization to determine the position/momentum split. The BRST approach, which is described in section 3, is a cohomological formulation which readily allows a quantization scheme which can be used for path integral quantization, provided this was the case for the unconstrained phase space. The situation described so far is a rather idealized, oversimplified one, most real physical systems involve quantum field theory rather than quantum mechanics, and the group action can be identified locally rather than globally. There is a discussion of these matters in appendix A, in the rest of this section we will continue to work in the idealized framework so that the key ideas can be presented in a simple manner.

A modification of this reduced phase space structure will now be described, which will lead in section 4 to a construction which aims to provide the appropriate modification of the

BRST procedure for a system with what is called reducible symmetry. This concept was first introduced and studied by Batalin and Vilkovisky [1, 2] in the Lagrangian formalism, and is discussed in section 5. A key feature is that there is only a partial symmetry of the system under the action of a Lie group. In the Hamiltonian framework described above, this means that only some of the constraints are satisfied. An important idea in the current paper is that the missing constraints can be incorporated into the formalism by introducing new variables. At this stage it is not clear whether the procedures described are applicable to all first-order reducible systems, further work is required here, but it is clear that the method described does provide a more fundamental account of the BRST operator in the reducible case, clarifying the role of 'ghosts for ghosts', and relating them to the even generators of $S(\mathfrak{g}\mathrm{d})$ which appear in the de Rham models of equivariant cohomology.

In the canonical setting, the ingredients of the systems to be considered again include a Hamiltonian action of a Lie group G on a symplectic manifold $\mathcal{N}$. Additionally, G has an Abelian subgroup H with a particular property, and the constraints take the form

$$T_a - \langle v, \xi_a \rangle = 0, \tag{4}$$

where v is an arbitrary element of $\mathfrak{h}^*$, the dual of the Lie algebra $\mathfrak{h}$ of H and T is the constraint map as before. (This form of the constraints is related to the Lagrangian approach to reducible symmetries in section 5.) Constraints of this form are possible if we extend the phase space $\mathcal{N}$ by taking the Cartesian product with T^*H.

The property required of H, which among other things ensures that $\mathfrak{h}^*$ can be uniquely identified as a subspace of $\mathfrak{g}^*$, is that there is a subspace $\mathfrak{k}$ of $\mathfrak{g}$ such that

$$\mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g} \qquad \text{and} \qquad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}. \tag{5}$$

(An example is when G is semi-simple and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.) This property means that $\mathfrak{h}^*$ can be identified as the subspace of $\mathfrak{g}^*$ whose elements u satisfy $\langle u, \xi \rangle = 0$ for all ξ in $\mathfrak{k}$. It will be useful to use a basis

$$\{\xi_\alpha, \xi_r | \alpha = 1, \ldots, l, r = 1 + l, \ldots, m\} \tag{6}$$

of $\mathfrak{g}$, with $\{\xi_\alpha | \alpha = 1, \ldots, l\}$ a basis of $\mathfrak{h}$ (l being the dimension of H) and $\{\xi_r | r = 1, \ldots, m-l\}$ a basis of $\mathfrak{k}$, and use the notational convention that Greek letters are used as indices for elements of bases of $\mathfrak{h}$ while Latin indices from the second half of the alphabet are used for $\mathfrak{k}$ and from the first half for $\mathfrak{g}$ as a whole. The only structure constants C^c_{ab} with respect to this basis which are non-zero are then those of the form $C^s_{\alpha r}$, C^t_{rs} and C^α_{rs}.

The extended phase space $\mathcal{N}' = \mathcal{N} \times T^*H$ has a symplectic form

$$\omega' = \omega + \mathrm{d}v_\alpha \wedge \mathrm{d}w^\alpha, \tag{7}$$

where the variables v_α are interpreted as coordinates on the cotangent space at each point of H, which is identified with $\mathfrak{h}^*$ and w^α are coordinates on H. (The summation convention for repeated indices is used except when explicitly stated to the contrary.) In terms of these coordinates the constraints take the form

$$v_\alpha - T_\alpha = 0, \quad \alpha = 1, \ldots, l \qquad \text{and} \qquad T_r = 0, \quad r = 1 + l, \ldots, m, \tag{8}$$

where $v_\alpha = \langle v, \xi_\alpha \rangle$ is a coordinate on the extended phase space $\mathcal{N}'$. These constraints do not in general form a closed algebra under the Poisson bracket, instead one has a partly second class system, neither do they correspond to a G action on $\mathcal{N}$ or $\mathcal{N}'$.

These problems stem from the fact that we have so far overlooked the fact that the variables v_α, $\alpha = 1, \ldots, l$ have canonical conjugates w^α which are also constrained. By taking this into account it will be shown that the reduced phase space for the system is in fact $\mathcal{N}' /\!\!/ (G \rtimes H)$, where $G \rtimes H$ denotes the semi-direct product of G and H corresponding to the action of H

on G by inverse conjugation and the action of $G \rtimes H$ on $\mathcal{N}'$ corresponds to the constraint map whose components with respect to bases $\{\xi_\alpha, \xi_r\}$ of $\mathfrak{g}$ and $\{\lambda_\alpha, \alpha = 1, \ldots, l\}$ of $\mathfrak{h}$ are respectively

$$T_\alpha, T_r \qquad \text{and} \qquad v_\alpha - T_\alpha. \tag{9}$$

(The Lie algebra of $G \rtimes H$ corresponds to that of the direct product $G \times H$ with additional non-zero brackets $[\lambda_\alpha, \xi_r] = -C^s_{\alpha r}\xi_s$.)

This reduced phase space will be obtained from the physical constraints (8) if $v_\alpha, \alpha = 1, \ldots, l$ are regarded as dynamical variables, with canonically conjugate momenta w^α. Since the original Lagrangian from which the constraints have been derived will not have had any dependence on the time derivative of v_α, additional constraints $w^\alpha = 0, \alpha = 1, \ldots, l$ must also be satisfied. Thus we have a system with $m + l$ primary constraints $T_r, v_\alpha - T_\alpha, w^\alpha, r = 1 + l, \ldots, m, \alpha = 1, \ldots, l$.

There are also secondary constraints: since the Hamiltonian of the system is independent of w^α, the equation of motion for v_α is simply $\dot{v}_\alpha = 0$, so that v_α is constant. We choose $v_\alpha = 0$, which then gives us l secondary constraints. The full set of constraints is thus

$$\{T_r, T_\alpha - v_\alpha, v_\alpha, w^\alpha, r = 1 + l, \ldots, m, \alpha = 1, \ldots, l\}. \tag{10}$$

From the Poisson bracket $\{v_\alpha - T_\alpha, w^\beta\} = -\delta^\beta_\alpha$ we see that the constraints $w^\alpha, v_\alpha - T_\alpha, \alpha = 1, \ldots, l$ form a second class set which reduces the extended phase space $\mathcal{N}'$ to $\mathcal{N}' /\!\!/ H$ under the action of H generated by $v_\alpha - T_\alpha$, which we can represent explicitly as the subspace of $\mathcal{N}'$, where $u^\alpha = 0$ and $v_\alpha - T_\alpha = 0$ for $\alpha = 1, \ldots, l$ with the symplectic form whose Poisson brackets are given by the Dirac bracket

$$\{f, g\}_D = \{f, g\} - \{f, v_\alpha - T_\alpha\}\{g, w^\alpha\} + \{g, v_\alpha - T_\alpha\}\{f, w^\alpha\}. \tag{11}$$

The remaining constraints $T_r, r = 1 + l, \ldots, m$ and $v_\alpha, \alpha = 1, \ldots, l$ form a first class set on this reduced space (where we can in fact replace v_α by T_α), and the corresponding reduction process then reduces this space further. This two stage reduction can be effected all at once by the action of $G \rtimes H$ as indicated above. A very comprehensive study of two stage reduction, with applications in a number of classical contexts, has been made by Marsden, Misiolek, Ortega, Perlmutter and Ratiu in [25, 26].

In the following section, the BRST quantization procedure for a closed constraint algebra will be described, while in section 4 it will be shown how this construction may be modified to take into account the reduced phase space of the kind just described, corresponding to an action of $G \rtimes H$ on an extended phase space.

3. The BRST procedure for a closed constraint algebra

In this section, we review the BRST procedure for the standard reduced phase space corresponding to a closed constraint algebra. The reduced phase space is the space $\mathcal{N} /\!\!/ G = C/G$ constructed in section 2, with $C = \phi^{-1}(0)$. The formulation of BRST cohomology in the canonical setting was first given by Henneaux [8] and by McMullan [28], providing a powerful development of the BFV construction of the vacuum generating functional of a gauge theory [1, 12, 17–19]. The BRST construction was expressed in a more abstract mathematical setting by Kostant and Sternberg [9] and by Stasheff [31, 32].

The idea is to construct a BRST operator Q whose zero degree cohomology theory agrees with the space of smooth functions $\mathcal{F}(\mathcal{N} /\!\!/ G)$ on the reduced phase space, and also to construct a super phase space so that the BRST operator Q is implemented by Poisson bracket. The exposition here largely follows [9]. The operator is constructed in two stages. First, we define a superderivation

$$\delta : \Lambda^q(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) \mapsto \Lambda^{q-1}(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N})$$

by its action on generators:

$$\delta(\pi \otimes 1) = 1 \otimes T_\pi, \qquad \delta(1 \otimes f) = 0, \tag{12}$$

where $\pi \in \mathfrak{g}$ and $f \in \mathcal{F}(\mathcal{N})$. It follows immediately that $\delta^2 = 0$. (This is the Koszul complex, as seems first to have been observed by McMullan.) Also $\mathrm{Ker}^0\,\delta = \mathcal{F}(\mathcal{N})$ while $\mathrm{Im}^0\,\delta = \delta(\mathfrak{g})\mathcal{F}(\mathcal{N})$. Now $\delta(\mathfrak{g})\mathcal{F}(\mathcal{N})$ is the ideal of $\mathcal{F}(\mathcal{N})$ consisting of functions which vanish on the constraint surface C, and the space of smooth functions on C can be identified with $\mathcal{F}(\mathcal{N})$ modulo this ideal. Thus $\mathcal{F}(C) \cong \mathrm{H}^0(\delta)$, and the first part of the construction of the BRST operator has been achieved.

To complete the construction, suppose that K is a $\mathfrak{g}$ module, and define the operator

$$\mathrm{d} : K \to \mathfrak{g}^* \otimes K$$

by setting $\langle \mathrm{d}k, \pi\rangle = \pi k$ for all π in $\mathfrak{g}$ and k in K. This operator can be extended to become a superderivation

$$\mathrm{d} : \Lambda^p(\mathfrak{g}^*) \otimes K \to \Lambda^{p+1}(\mathfrak{g}^*) \otimes K$$

by defining $\mathrm{d}\eta$ for η in $\mathfrak{g}^*$ to be the exterior derivative of η regarded as a left invariant one form on G. (This, as observed by Stasheff, is the standard Chevalley–Eilenberg differential for the Lie algebra cohomology of G.) Using the fact that d on $\mathfrak{g}^*$ is the transpose of the bracket on $\mathfrak{g}$, it can be shown that $\mathrm{d}^2 = 0$. Also, it follows from the definition that $\mathrm{Ker}^0\,\mathrm{d}$ is equal to the set $K^{\mathfrak{g}}$ of $\mathfrak{g}$ invariants in K while $\mathrm{Im}^0\,\mathrm{d}$ is zero. Thus $\mathrm{H}^0\mathrm{d}$ is equal to $K^{\mathfrak{g}}$.

If we now set $K = \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N})$, with the $\mathfrak{g}$ action on K defined by

$$\begin{aligned}\xi(\pi_1 \wedge \cdots \wedge \pi_q \otimes f) = &\sum_{r=1}^{q} \pi_1 \wedge \cdots \wedge \pi_{r-1} \wedge [\xi, \pi_r] \wedge \pi_{r+1} \wedge \cdots \wedge \pi_q \otimes f \\ &+ \pi_1 \wedge \cdots \wedge \pi_q \otimes \{T_\xi, f\},\end{aligned} \tag{13}$$

then the $\mathfrak{g}$ action commutes with the action of δ on K, so that δ and d commute and d is well defined on the δ cohomology groups of K. Thus $\mathrm{H}^0(\mathrm{H}^0(\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N})))$ is well defined and equal to the $\mathfrak{g}$ invariant elements of $\mathcal{F}(C)$, and thus to $\mathcal{F}(\mathcal{N} /\!\!/ G)$.

The properties of the two differentials may be summarized in the diagram

$$\begin{array}{ccc}\Lambda^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) & \stackrel{\delta}{\to} & \Lambda^p(\mathfrak{g}^*) \otimes \Lambda^{q-1}(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) \\ \mathrm{d}\downarrow & & \\ \Lambda^{p+1}(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) & & \end{array}$$

and we have a double complex

$$\mathrm{D} : \Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) \to \Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}).$$

with $\mathrm{D} = \mathrm{d} + (-1)^p\delta$. If we define the total degree of an element of $\Lambda^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N})$ to be $p - q$, then D raises degree by 1. Under certain technical assumptions [9] $\mathrm{H}^0\mathrm{D}$ is equal to $\mathrm{H}^0(\mathrm{H}^0(\Lambda\,\mathfrak{g}^* \otimes \Lambda\,\mathfrak{g} \otimes \mathcal{F}(\mathcal{N})))$, so that we have constructed a complex whose zero cohomology is equal to $\mathcal{F}(\mathcal{N} /\!\!/ G)$, in other words to the observables on the true phase space of the system.

If we now construct the $(2n, 2m)$-dimensional symplectic supermanifold

$$\mathcal{S}\mathcal{N} = \mathcal{N} \times \mathbb{R}^{0,2m}$$

with symplectic form $\omega + \mathrm{d}\pi_a \wedge \mathrm{d}\eta^a$, where $\pi_a, \eta^a, a = 1, \ldots, m$ are natural coordinates on the $\mathbb{R}^{0,2m}$ factor, then $\mathcal{F}(\mathcal{SN}) \cong \Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N})$ and D can be realized by taking the Poisson bracket with the function

$$Q = \eta^a T_a - \tfrac{1}{2} C^c_{ab} \eta^a \eta^b \pi_c.$$

As a result the Poisson brackets with respect to the symplectic form $\omega + \mathrm{d}\pi_a \wedge \mathrm{d}\eta^a$ close on the zero cohomology of D and correspond to the Poisson brackets on the reduced phase space. Quantization of this system is straightforward, given a quantization on the original unconstrained phase space $\mathcal{N}$. The Hilbert space of states is taken to be $\mathcal{H} \otimes \mathcal{F}(\mathbb{R}^{0,m})$, where $\mathcal{H}$ is the space of states for $\mathcal{N}$. A typical element is $f_{a_1\ldots a_p}\eta^{a^1}\cdots\eta^{a^p}$ where each $f_{a_1\ldots a_p}$ is in $\mathcal{H}$ and $\eta^a, a = 1, \ldots, m$ are natural coordinates on $\mathbb{R}^{0,m}$. (If the G action is local, in the manner described in appendix A, then the super phase space and the space of states may be twisted products rather than simply Cartesian products, and the operators δ and d defined locally but in a globally consistent manner.)

Observables on the super phase space will be operators of the form

$$A^{b_1\ldots b_q}_{a_1\ldots p}\eta^{a^1}\ldots\eta^{a^p}\pi_{b_1}\ldots\pi_{b_q}$$

where each $A^{b_1\ldots b_q}_{a_1\ldots a_p}$ is an observable on $\mathcal{N}$. The observables η^a, which are known as 'ghosts', are represented on states as multiplication operators, while the ghost momenta π_b are represented by

$$\pi_b = -\mathrm{i}\frac{\partial}{\partial\eta^b}. \tag{14}$$

An obvious but important consequence of this scheme is that the quantized BRST operator Q has square zero. We can thus implement the BRST cohomology at the quantum level by defining physical observables to be observables which commute with Q, the quantized BRST operator, modulo observables which are themselves commutators with Q. Physical states are then defined to be states annihilated by Q modulo those in the image of Q. (Further aspects of BRST quantum dynamics, including the gauge fixing necessary in the path integral approach, are described in [22, 29, 30].)

4. The modified procedure for a class of open constraint algebras

In this section, we construct the analogue of the BRST procedure for the case where the constraints and reduced phase space are those of section 2, corresponding to a one-dimensional subgroup H of our m-dimensional symmetry group G. The operator will be expressed in a form that allows gauge fixing and path-integral quantization as in [10, 29].

Proceeding directly with the $G \rtimes H$ action on $\mathcal{N}' = \mathcal{N} \times T^*(H)$ with constraint map whose components are the constraint functions T_α, T_r and $v_\alpha - T_\alpha = 0$, we obtain the BRST operator

$$Q = \eta^a T_a + \theta(v_\alpha - T_\alpha) - \tfrac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c + \theta^\alpha\eta^r C^s_{\alpha r}\pi_s \tag{15}$$

acting on the space $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}') \otimes \Lambda(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h})$. (Here as in section 3 we use π_α, η^a for elements of $\mathfrak{g}$ and $\mathfrak{g}^*$, while for the copy of $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ corresponding to the second factor in $G \rtimes H$ we use ρ_α and θ^α). If we define the function $\mathcal{L}_\alpha$ by

$$\mathcal{L}_\alpha = \left\{\eta^a T_a - \tfrac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c, \pi_\alpha\right\} = T_\alpha - C^s_{\alpha r}\eta^r\pi_s, \tag{16}$$

the BRST operator can be expressed in the form

$$Q = \eta^a T_a - \tfrac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c - \theta^\alpha(\mathcal{L}_\alpha - v_\alpha). \tag{17}$$

This operator can be expressed as the sum of two commuting operators $Q_G = \eta^a T_a - \frac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c$ and $Q_H = -\theta^\alpha(\mathcal{L}_\alpha - v_\alpha)$. (Each of these two parts is canonical, independent of any choice of basis of $\mathfrak{g}$ or $\mathfrak{h}$.) The cohomology corresponding to Q_H is acyclic, and explicitly solved by taking functions which are invariant under ρ_α and $\mathcal{L}_\alpha - v_\alpha$. If we make the Kalkman transformation [4], that is, we conjugate by $\exp(-\theta^\alpha\pi_\alpha)$, we see that an equivalent cohomology is that of

$$Q = \eta^a T_a - \tfrac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c + \theta^\alpha v_\alpha, \tag{18}$$

where the auxiliary conditions are now $\rho_\alpha = \pi_\alpha$ and $\mathcal{L}_\alpha - v_\alpha = 0$. (This transformation is the analogue of an extension due to Kalkman [4] of the Mathai–Quillen isomorphism [27] used in equivariant de Rham theory.)

This is one possible formulation of the BRST operator of the theory. We will now use some techniques from equivariant de Rham theory, which are also valid in this context, to give an alternative formulation which allows quantization including a method for implementing the auxiliary conditions by gauge fixing and corresponds to the BRST operator for reducible symmetries. Some terminology is required, which is summarized in appendix B, following the book of Guillemin and Sternberg [21] where more details may be found.

The space $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N})$ can be given the structure of an H^* algebra (definition B.1): the $\tilde{\mathfrak{h}}$ action is defined by setting i_α to act as a Poisson bracket with π_α, d to be the BRST operator Q_G and L_α to act as a Poisson bracket with $\mathcal{L}_\alpha = T_\alpha + C^s_{\alpha r}\eta^r\pi_s$, while the H action is defined to be the adjoint action on $\mathfrak{g}$, the co-adjoint action on $\mathfrak{g}^*$ and the original H action on $\mathcal{N}$ with constraint map T_α. It can further be given the structure of a W^* module (definition B.1) if multiplication by the generator odd generator κ^α of W is given by the Poisson bracket with η^α and multiplication by the even generator u^α is given by the Poisson bracket with $d\eta^\alpha = -\frac{1}{2}C^\alpha_{ab}\eta^a\eta^b$.

Another W^* module F may be defined by setting

$$F = \mathcal{F}(T^*(H)) \otimes \Lambda(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h}) = \mathcal{F}(T^*(H) \times \mathbb{R}^{0,2l}). \tag{19}$$

Taking coordinates $v_\alpha, w^\alpha, \theta^\alpha$ and ρ_α on $T^*(H) \times \mathbb{R}^{0,2l}$ as before, the H^* structure of F is defined by letting $i_{F\alpha}$ act as a Poisson bracket with $-\rho_\alpha$, $L_{F\alpha}$ with $-v_\alpha$ and d_F with $\sum_{\alpha=1}^{l}\theta^\alpha v_\alpha$, while H acts trivially on $\mathfrak{h}$ and $\mathfrak{h}^*$ and naturally on $T^*(H)$. The W action on F is defined by letting the generator κ^α of $\mathfrak{h}^*$ act as a Poisson bracket with θ^α, while the generators u^α of W act by the Poisson bracket with w^α. The basic cohomology of $Q \otimes 1 + 1 \otimes D_F$ on $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) \otimes F$, that is, the cohomology on the subspace of $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) \otimes F$ whose elements have zero Poisson bracket with $\mathcal{L}_\alpha - v_\alpha$ and with $\pi_\alpha - \rho_\alpha$, is then the BRST cohomology of the doubly reduced phase space in the form (18). By theorem B.2, if we take E to be the Weil algebra $S(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h}^*)$ of H we see that an alternative form of the BRST cohomology is the basic cohomology of

$$\eta^a T_a - \tfrac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c + u^\alpha\rho_\alpha, \tag{20}$$

a form which suggests a close analogy with equivariant de Rham cohomology. (In this case the basic conditions are zero Poisson bracket with $\mathcal{L}_\alpha$ and with $\pi_\alpha + \rho_\alpha$.) A gauge-fixing procedure which implements these basic conditions is constructed in [30].

A further possibility is the Cartan model, constructed by taking the Kalkman transformation as before, which this time gives the BRST operator in the form

$$\eta^a T_a - \tfrac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c + \theta^\alpha\mathcal{L}_\alpha + u^\alpha\rho_\alpha - u^\alpha\pi_\alpha, \tag{21}$$

with basic condition $\mathcal{L}_\alpha = 0, \rho_\alpha = 0$ which is the same as the cohomology of $\eta^a T_a - \frac{1}{2}\eta^a\eta^b C^c_{ab}\pi_c - u^\alpha\pi_\alpha$ on $\mathcal{L}_\alpha$ invariant elements of $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathcal{F}(\mathcal{N}) \otimes S(\mathfrak{h}^*)$.

5. Reducible symmetry

In this section, the notion of reducible symmetry as introduced by Batalin and Vilkovisky [1, 2] is related to the constrained systems whose reduced phase space and BRST quantization is considered in sections 2 and 4. The aim is, using somewhat informal terminology based on the Lagrangian approach, to show how the particular class of the constrained system studied in this paper relates to the notion of reducible symmetry. As already remarked, the work of Batalin and Vilovisky on systems with reducible symmetries included the development of what has become known as BV quantization, an extension of the BRST technique, which gave a consistent functional integral expression for the vacuum expectation value of theories with reducible constraints (and has additionally led to a number of interesting developments in mathematics and physics, involving the master equation and odd symplectic manifolds, which are not considered in this paper). These methods were further studied by Fisch, Henneaux, Stasheff and Teitelboim [3] who gave an algebraic analysis of the BRST operator in terms of Koszul–Tate resolutions. A full account of these ideas may be found in the book of Henneaux and Teitelboim [22], while an example of such a system is considered in section 6.

For simplicity, so that the basic algebraic features are clear, we will restrict the discussion at this stage to a quantum mechanical system where the symmetry of the system corresponds to a finite-dimensional group G acting on the fields $x^i(t), i = 1, \ldots, n$ of the system. (The appendix shows how this may be extended to some infinite-dimensional group actions.) Suppose that corresponding to a basis $\{\xi_a | a = 1, \ldots, m = \dim G\}$ of the Lie algebra $\mathfrak{g}$ of G the infinitesimal action of the group element $1 + \sum_{a=1}^m \epsilon^a \xi_a$ is

$$\delta_\epsilon x^i = \sum_{a=1}^m \epsilon^a(t) R_a{}^i(x), \tag{22}$$

where the $R_a{}^i$ satisfy

$$R_a{}^j \frac{\partial}{\partial x^j}\left(R_b{}^i\right) - R_b{}^j \frac{\partial}{\partial x^j}\left(R_a{}^i\right) = \sum_{c=1}^m C^c_{ab} R_c{}^i, \tag{23}$$

with C^c_{ab} being the structure constants of $\mathfrak{g}$ as before. If the action of a system is

$$S(x(\cdot)) = \int \mathrm{d}t\, \mathfrak{L}(x^i(t), \dot{x}^i(t)) \tag{24}$$

and

$$\sum_{i=1}^n R_a{}^i \frac{\delta \mathfrak{L}}{\delta x^i} = 0 \qquad \text{for} \quad a = 1, \ldots, m, \tag{25}$$

where $\frac{\delta \mathfrak{L}}{\delta x^i} = \frac{\partial \mathfrak{L}}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}^i}\right)$, then the action is invariant under the action of G and there are conserved Noether currents

$$J_a = R_a{}^i \frac{\partial \mathfrak{L}}{\partial \dot{x}^i} \tag{26}$$

which under Legendre transformation becomes constraints

$$T_a = R_a{}^i p_i = 0, \tag{27}$$

where $p_i, i = 1, \ldots, n$, are the conjugate momenta of x^i. These constraints are first class and obey the algebra

$$\{T_a, T_b\} = \sum_{c=1}^m C^c_{ab} T_c. \tag{28}$$

As before we assume that the number n of fields is at least as large as the dimension m of G.

If the m vectors R_a are linearly independent for all x, or equivalently the matrix $\left(R_a{}^i\right)$ has rank m, the system is said to have an *irreducible* symmetry, and the BRST procedure described in section 3 is applicable. A *reducible* symmetry occurs when the matrix has rank $m - l$, with $l > 0$, in which case the group property (23) of the infinitesimal transformations will not in general be satisfied. The concept of reducible symmetry was first identified by Batalin and Vilkovisky [1, 2]. In this paper, we are concerned with reducible systems for which the infinitesimal transformations $R_a{}^i$ are of the form

$$R_a{}^i = \sum_{b=1}^{m} M_a{}^b U_b{}^i, \tag{29}$$

where $\left(M_a{}^b\right)$ is an $m \times m$ matrix of rank $m - l$ with $0 < l < m$ and the elements $U_b{}^i$ have maximal rank m and do satisfy the group property, that is,

$$U_a{}^j \frac{\partial}{\partial x^j}\left(U_b{}^i\right) - U_b{}^j \frac{\partial}{\partial x^j}\left(U_a{}^i\right) = \sum_{c=1}^{m} C^c_{ab} U_c{}^i. \tag{30}$$

(It may be conjectured that all reducible symmetries take this form.) As a result there are l non-trivial linear relations of the form

$$\sum_{a=1}^{m} \lambda^a_\alpha R_a = 0 \qquad \alpha = 1, \ldots, l, \tag{31}$$

where R_a denotes the vector $\left(R_a{}^i\right)$ so that there are of course only $m - l$ independent transformations, which will not in general form a Lie algebra. (Recent discussions of Noether's second theorem [24, 33] consider related issues.) By a suitable choice of basis we can set

$$R_\alpha = 0, \quad \alpha = 1, \ldots, l \qquad \text{and} \qquad R_r = U_r \quad r = 1 + l, \ldots, m. \tag{32}$$

On passing to the Hamiltonian formulation of the system, there will be $m - l$ constraints

$$T_r \equiv U_r{}^i p_i = 0 \qquad r = 1 + l, \ldots, m, \tag{33}$$

which in general will not form a first class system. Corresponding to the 'missing' Noether currents $U_\alpha, \alpha = 1, \ldots, l$ there are functions T_α which are not constrained to be zero. This leads to the modified reduction process described in section 2, involving an extended phase space, and thence to the modified BRST quantization constructed in section 4 which is equivalent to the BRST operator obtained by the algebraic techniques of BV quantization [1–3, 22].

Chemla and Kalkman [5] have shown that the BRST operator for certain topological theories corresponds to that for a system of reducible symmetries, using the transformation $\exp(\theta^\alpha \pi_\alpha)$ which we also use in this paper. We have derived this result in a more general context directly from the constraints of the system. Outstanding questions include analysing whether all reducible symmetries lead to constrained systems of this nature. In the following section, we give an example of the equivariant BRST quantization of a particular system.

6. An example

As an example of the structures described in sections 2 and 4, a topological model will now be described. The setting of this model is an n-dimensional Riemannian manifold $\mathcal{M}$ with metric g on which there is an isometric $U(1)$ action generated by a Killing vector X. The classical action for this model is

$$S(x(.)) = \int_0^t v x^* \tilde{X}, \tag{34}$$

where $x : [0, t] \to \mathcal{M}$ is a path in $\mathcal{M}$, $\tilde{X}$ is the one form dual to X via g and v is a constant. Using local coordinates x^i, $i = 1, \ldots, n$ on $\mathcal{M}$ the action takes the form

$$S(x(.)) = \int_0^t vX_i(x(t))\dot{x}^i(t)\,\mathrm{d}t, \tag{35}$$

where $X_i = g_{ij}X^j$ are the components of $\tilde{X}$. The variational derivative of the Lagrangian $\mathfrak{L}(x, \dot{x}) = vX_i(x)\dot{x}^i$ is

$$\frac{\delta\mathfrak{L}}{\delta x^i} = 2v\mathcal{D}_j X_i \dot{x}^j = -2v\mathcal{D}_i X_j \dot{x}^j, \tag{36}$$

where $\mathcal{D}$ denotes covariant differentiation with respect to the Levi-Civita connection of the metric g, so that the Lagrangian is invariant under infinitesimal transformations $\delta x^i = \epsilon^i$ if ϵ^i satisfies $\epsilon^i X_i = 0$. This gives n reducible symmetries with one linear-dependence relation. (It is here that the importance of X being a Killing vector first appears, since this ensures that $\frac{1}{2}(\partial_i X_j - \partial_j X_i) = \mathcal{D}_i X_j = -\mathcal{D}_j X_i$.)

Proceeding to the Euclidean time Hamiltonian formalism, the conjugate momentum to x^i is

$$p_i = \mathrm{i}vX_i, \tag{37}$$

and we see that the system has n first class constraints $T_i \equiv p_i - \mathrm{i}vX_i = 0$, which are of the general form (8) but in a geometrically natural basis rather than an $\mathfrak{h}, \mathfrak{k}$ basis. It is also necessary to regard v as a dynamical variable rather than a constant. (Whether this is a general feature of the constraints of a system with reducible symmetry is a question which needs further exploration.) As expected the constraints of this system do not form a closed algebra; using the standard symplectic form in the phase space $T^*\mathcal{M}$ gives

$$\{T_i, T_j\} = 2\mathrm{i}v\mathcal{D}_i X_j. \tag{38}$$

This situation corresponds to that considered in sections 2, 4 and 5 (in the local version described in appendix A) with $\tilde{G}$ the diffeomorphism group of $\mathcal{M}$ and H the group $U(1)$ acting on $\mathcal{M}$. The group G which acts locally is then, as in example A.1, the n-dimensional translation group $\mathbb{R}^n$ with constraint map $T_i = p_i$. To see that the Lie algebra of H has the required property, let y be a point in $\mathcal{M}$ where X is not zero and $\{X\} \cup \{\xi_r | r = 2, \ldots, n\}$ be a basis of the tangent space at y with each ξ_r orthogonal to X. Then, because X is a Killing vector, it can be shown that $[\xi_r, X]$ is also orthogonal to X. Thus, if we identify $\mathfrak{k}$ as the span of $\{\xi_r | r = 2, \ldots, n\}$ and $\mathfrak{h}$ as the span of $\{X\}$ we see that $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$ as required.

In this case we have an S^1 extension, so that the modified BRST procedure of section 4 gives as a BRST operator

$$Q = \eta^i p_i + u\rho, \tag{39}$$

with auxiliary conditions $\rho = \pi$ and $\mathcal{L} = 0$. (Since H is one-dimensional we drop the index α.) On quantization we obtain the differential in the Weil model of equivariant cohomology of $\mathcal{M}$ under the $U(1)$ action generated by X. The full quantization of this model, using the Kalkman form [4], including gauge fixing, has been described in [30]. The model constructed in this section is equivalent to that constructed by Witten [10], as can be shown by integrating out the u, v, θ and ρ variables or by conversion to the Cartan model. It is also the same as that obtained by Chemla and Kalkman [5]. The derivation given in this section shows how the model can be understood as the BRST quantization of a simple classical model, with the conditions $\rho = \pi$ and $\mathcal{L} = 0$ emerging from the physics.

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Appendix A. The local group action and the non-uniqueness of the constraints

In section 2, the standard approach to the reduced phase space of a constrained Hamiltonian system was described, together with a modification for a more general set of constraints than a standard first class set.

Even in the standard setting we have glossed over a difficulty which seems so far to have received a rather incomplete treatment in the literature. This relates to the non-uniqueness of the constraint functions used to define the constraint submanifold, and hence the reduced phase space. This non-uniqueness is a simple consequence of the fact that the constraints can be multiplied by arbitrary nowhere-zero functions and still define the same constraint submanifold, and satisfy a closed constraint algebra, although this will vary with the choice of functions, and generally not form a finite-dimensional Lie algebra. More fundamentally, it relates to the fact that it is gauge or local symmetries which lead to the constrained Hamiltonian systems discussed in this paper.

In this appendix, we address these issues by describing the reduction process for the case where the action of the finite group G can only be identified locally, although there is a group $\tilde{G}$ (of infinite dimension) which acts globally on $\mathcal{N}$. It will emerge that essentially the same construction of a reduced phase space can be made in this more general setting provided that the $\tilde{G}$ action has properties which we will now define. (In the example below G is a diffeomorphism group, and we also have in mind the situation where $\tilde{G}$ is the group of automorphisms of a fibre bundle, a so-called gauge group, but it is possible that even more general situations may occur, and so the properties required are those that are essential.) Suppose that $\tilde{G}$ is a Lie group which acts symplectically on $\mathcal{N}$ and that there is an open cover $\{U_\sigma | \sigma \in \Lambda\}$ of $\mathcal{N}$ and, for each σ in Λ, a neighbourhood V_σ of the identity of G such that V_σ acts locally on U_σ in the following sense: there is a map $V_\sigma \times U_\sigma \to \mathcal{N}$, $(g, y) \to gy$ such that if g, h and gh are in V_σ and y, hy are in U_σ then $(gh)y = g(hy)$. It is also required that the local G action is free, although the global $\tilde{G}$ action may have fixed points. Also suppose that this local G action is compatible with the $\tilde{G}$ action in that if $\tilde{\eta}$ is an element of $\tilde{\mathfrak{g}}$, the Lie algebra of $\tilde{G}$, then, for each basis $\{\xi_a | a = 1, \dots, m\}$ of $\mathfrak{g}$ (the Lie algebra of the finite group G) and each $\sigma \in \Lambda$ there exist m functions $q^a_{\tilde{\eta}\sigma} : U_\sigma \to \mathbb{R}$, $a = 1, \dots, m$ such that for every f in $\mathcal{F}(\mathcal{N})$

$$\underline{\tilde{\eta}} f \big|_{U_\sigma} = q^a_{\tilde{\eta}\sigma} \underline{\xi_a} f \big|_{U_\sigma}. \tag{A.1}$$

(Here we again use the notation that $\underline{\tilde{\eta}}$ denotes the vector field on $\mathcal{N}$ corresponding to the element $\tilde{\eta}$ of $\tilde{\mathfrak{g}}$, and so on.)

This group action is said to be Hamiltonian if both the global $\tilde{G}$ action and the local G action have constraint maps, denoted $\tilde{T}$ and T_σ respectively, with

$$\tilde{T}_{\tilde{\eta}}|_{U_\sigma} = q^a_{\tilde{\eta}\sigma} T_{\sigma\, a}. \tag{A.2}$$

The number of independent constraints is equal to the dimension of G rather than $\tilde{G}$.

An example of this structure will now be described.

Example A.1. Suppose that $\mathcal{N}$ is the cotangent bundle $T^*\mathcal{M}$ of an n-dimensional manifold $\mathcal{M}$, $\tilde{G}$ is the diffeomorphism group of $\mathcal{M}$ (which acts naturally on $T^*\mathcal{M}$) and G is the n-dimensional translation group $\mathrm{Tr}(n)$. (As a manifold this group is simply $\mathbb{R}^n$.) We construct the open cover $\{U_\sigma | \sigma \in \Lambda\}$ of $T^*\mathcal{M}$ from an open cover $\{W_\sigma | \sigma \in \Lambda\}$ of $\mathcal{M}$ by coordinate neighbourhoods, setting $U_\sigma = T^*W_\sigma$. We then define the local action of $G = \mathrm{Tr}(n)$ by $(x^i, p_j) \to (x^i + t^i, p_j)$, where x^i, $i = 1, \dots, n$ are local coordinates on W_σ, (x^i, p_i) are the corresponding local coordinates on T^*W_σ and t^i, $i = 1, \dots, n$ is a sufficiently small element of $\mathbb{R}^n$. The local constraint maps for the G action are $T_i = p_i$. The Lie algebra of the diffeomorphism group of $\mathcal{M}$ may be identified with the set of vector fields on $\mathcal{M}$. If Y is a vector field on $\mathcal{M}$ with local coordinate expression $Y = Y^i \frac{\partial}{\partial x^i}$, then the global constraint map for η is

$$\tilde{T}_Y = Y^i p_i. \tag{A.3}$$

The two-stage process leading to the reduced phase space can be carried out as before; in the case where $\mathcal{N}$ has dimension $2n$ and the local group G has dimension m, the reduced phase space will have dimension $2(n-m)$. In terms of constraints, by proceeding to the larger group $\tilde{G}$, and allowing for the possibility of a local rather than global action by the finite group G, we have explained the observed multiple possibilities both for the set of constraints and for the algebra they form [22].

The modified reduced phase space corresponding to a subgroup H of G can also be handled in this more general setting. The requirement is a finite-dimensional subgroup H of the global group which locally has the properties (5). We can locally define the reduced phase space as before, except that there will be singularities at fixed points of H. Since (as will emerge from the example in section 6) we can construct a non-singular BRST operator even in this situation, following the procedure which would be valid were there no fixed points, we will regard the BRST quantization scheme as the more fundamental object, and not pause to consider a fuller definition of the reduced phase space in this context.

Appendix B. The Weil algebra of *H* and related constructions

In this appendix, we gather some definitions and a theorem from equivariant de Rham theory, using the book of Guillemin and Sternberg [21] where more details can be found.

Definition B.1. *Given an Abelian one-dimensional Lie group H with Lie algebra $\mathfrak{h}$,*

(a) the super Lie algebra $\tilde{\mathfrak{h}}$ is defined to be the algebra $\tilde{\mathfrak{h}} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1$ where $\mathfrak{h}_{-1}$ is a one-dimensional vector space with basis $\mathrm{i}_1, \ldots, i_l$, $\mathfrak{h}_{-1}$ is a one-dimensional vector space with basis $L_1, \ldots, L_l$ and $\mathfrak{h}_1$ is one-dimensional with basis d and all Lie brackets are trivial except $[d, i_\alpha] = L_\alpha$.

*(b) A H^* module is a super vector space A together with a linear representation of ϕ of H on A and a homomorphism of $\tilde{\mathfrak{h}} \to \mathrm{End}\, A$
which obey the consistency conditions:*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\phi(\exp(t\xi))\bigg|_{t=0} = L_\xi, \\ &\phi(a)L_\xi\phi(a^{-1}) = L_{\mathrm{Ad}_a\xi}, \\ &\phi(a)i_\xi\phi(a^{-1}) = i_{\mathrm{Ad}_a\xi}, \\ &\phi(a)\,\mathrm{d}\phi(a^{-1}) = \mathrm{d}. \end{aligned} \tag{B.1}$$

(c) The Weil algebra W of the group H is the algebra $S(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h}^*)$. The super algebra $\tilde{\mathfrak{h}}$ acts on W by superderivations with the only non-zero action on generators given by

$$\begin{aligned} &i_\alpha(1 \otimes \kappa^\alpha) = 1 \otimes 1 \\ &\mathrm{d}(1 \otimes \kappa^\alpha) = u^\alpha \otimes 1, \end{aligned} \tag{B.2}$$

where κ^α are generators of $\Lambda(\mathfrak{h}^*)$ and u^α are generators of $S(\mathfrak{h}^*)$.

(d) A W^* module for the group H is an H^* module E which is also a W module, with the map $W \otimes E \to E$ a morphism of H^* modules.

(e) *Corresponding algebra structures are defined when the vector spaces are algebras, H acts by automorphisms and $\tilde{\mathfrak{h}}$ by superderivations.*

A key theorem in equivariant de Rham theory is also valid in the form stated here. It allows the construction of alternative models of the equivariant BRST cohomology. The proof may be found in [21].

Theorem B.2. *Suppose that E and F are acyclic W^* algebras and that A is a W^* module. Then the cohomology of basic elements of $A \times E$ with respect to $D_A \otimes 1 + 1 \otimes D_E$ is the same as that of basic elements of $A \times F$ with respect to $D_A \otimes 1 + 1 \otimes D_F$, where an element of $A \otimes E$ or $A \otimes F$ is said to be basic if it is annihilated by both $i_\alpha \otimes 1 + 1 \otimes i_\alpha$ and $L_\alpha \otimes 1 + 1 \otimes L_\alpha$.*

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